# 2019-Sep-26 (lecture) MATB44-2019 Prof. Yun William Yu

Thursday, September 26, 2019 10:06 AM

## Outline:

- Proof the Banach fixed pt thm
- Multidimensional Picard iteration
- Statement of Picard–Lindelöf Thm

## Last time:

- We gave examples of contractions, including Picard iteration.
- To finish the existence–uniqueness argument, we need to prove the contraction principle, describe Picard iteration in systems, and show that the last is a contraction.

**Thm** (Tesch 2.1) : Banach fixed pt thm (contractive principle)

Let $C$ be a (nonempty) closed subset of a Banach space $X$ and let $K: C \to C$ be a contraction. Then $K$ has a unique fixed pt $\bar{x} \in C$ s.t.

$$\| K^n(x) - \bar{x} \| \leq \frac{\theta^n}{1-\theta} \| K(x) - x \|, \quad x \in C.$$

**proof.** **Existence** Fix $x_0 \in C$ and consider the sequence

$$x_n = K^n(x_0).$$

Then we have

$$\begin{aligned}
\| x_{n+1} - x_n \| &\leq \theta \| x_n - x_{n-1} \| \\
&\leq \theta^2 \| x_{n-1} - x_{n-2} \| \\
&\;\;\vdots \\
&\leq \theta^n \| x_1 - x_0 \|.
\end{aligned}$$

By the triangle inequality, for $n > m$

$$\begin{aligned}
\| x_n - x_m \| &= \| (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots (x_{m+1} - x_m) \| \\
&\leq \| x_n - x_{n-1} \| + \cdots + \| x_{m+1} - x_m \| \\
&\leq \theta^{n-1} \| x - x_0 \| + \cdots + \theta^m \| x_1 - x_0 \|
\end{aligned}$$

$n-m-1$

$$= \theta^m \sum_{j=0}^{n-m-1} \theta^j \|x_1 - x_0\|$$

$$= \theta^m \cdot \frac{1-\theta^{n-m}}{1-\theta} \|x_1 - x_0\| \le \frac{\theta^m}{1-\theta} \|x_1 - x_0\|.$$

Because for any $\varepsilon$, we can find $M$ s.t. $\forall n, m \ge M$, $\|x_n - x_m\| \le \varepsilon$, this sequence is Cauchy.

Because we are in a Banach space, Cauchy sequences have limits, which we will call $x_n \to \bar{x}$.

Note $\|K(\bar{x}) - \bar{x}\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, so $\bar{x}$ is a fixed pt.

**Uniqueness:** Let $\bar{x}$ and $\bar{y}$ be fixed points of $K$, $\bar{x}, \bar{y} \in C$.

Then $\|K(\bar{x}) - K(\bar{y})\| = \|\bar{x} - \bar{y}\|$.

But $\|K(\bar{x}) - K(\bar{y})\| \le \theta \|\bar{x} - \bar{y}\|$ because $K$ a contraction.

So $\|\bar{x} - \bar{y}\| \le \theta \|\bar{x} - \bar{y}\|$, $\theta < 1$.

$\Rightarrow \|\bar{x} - \bar{y}\| = 0 \Rightarrow \bar{x} = \bar{y}$. $\square$

## Picard iteration on system of ODEs

**Last time:** $\dot{x} = f(t,x)$, $x(t_0) = x_0$, $x : \mathbb{R} \to \mathbb{R}$.

Let $K : C(U, \mathbb{R}) \to C(U, \mathbb{R})$, $U \subseteq \mathbb{R}^2$ an open set.

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Then we had Picard iterates

$x_0 = x_0$

$x_1 = K(x_0)$

$\vdots$ converge to the solution $x(t)$

$x_1 = K(x_0)$

$\vdots$ converge to the solution $x(t)$.

**This time:** $\dot{x} = f(t,x)$, $x(t_0) = x_0$, $x : \mathbb{R} \to \mathbb{R}^n$

Let $K : C(U, \mathbb{R}^n) \to C(U, \mathbb{R}^n)$, $U \subseteq \mathbb{R}^{n+1}$ an open set.

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Then the sequence $x_0(t) = x_0$, $x_m = K(x_{m-1})$ converges to the solution $x$.

All we did is replace a scalar valued $x$ with a vector valued function. Or, equivalently, a system of 1st-order ODEs.

**Ex.** Say we have a first-order system of 2 ODEs.

I am going to use **bold green** for vectors to keep things clear, just for this example.

Let $\dot{\mathbf{x}}(t) = \begin{pmatrix} t x_2 \\ x_1 x_2 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_0^t t \cdot 1\, dt \\ \int_0^t 1\, dt \end{pmatrix} = \begin{pmatrix} 1 + \frac{t^2}{2} \\ 1 + t \end{pmatrix}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_0^t t(1+t)\, dt \\ \int_0^t (1+\frac{t^2}{2})(1+t)\, dt \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{t^2}{2} + \frac{t^3}{3} \\ 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8} \end{pmatrix}$$

Or equivalently

$\frac{dy}{dt} = tz$, $\frac{dz}{dt} = yz$

$y(0) = 1$ $\quad z(0) = 1$

$y_1(t) = 1 + \int_0^t t\, dt = 1 + \frac{t^2}{2}$

$z_1(t) = 1 + \int_0^t 1\, dt = 1 + t$

$y_2(t) = 1 + \int_0^t t(1+t)\, dt = 1 + \frac{t^2}{2} + \frac{t^3}{3}$

$z_2(t) = 1 + \int_0^t (1 + \frac{t^2}{2})(1+t)\, dt = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8}$.

See Tenenbaum 57.35 for full arithmetic

**Notice:** Picard iteration also works on systems of 1st-order ODEs. Recall that we can convert any system of higher-order ODEs into a system of first-order ODEs

Recall that we can convert any system of higher-order ODEs into a system of first-order ODEs.

So all we have to do now is prove when Picard iteration is a contraction, leading to the most important theorem of MATB44.

**Theorem: (Picard-Lindelöf):**
**(Tesch 2.2)**

Suppose $f: C(U, \mathbb{R}^n)$, where $U$ is an open subset of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. If $f$ is locally Lipschitz continuous in the 2nd argument, uniformly with respect to the 1st argument, then there exists a unique local solution $\bar{x}(t) \in C^1(I)$ of the the initial value problem, where $I$ is a closed interval around $t_0$.

**Define:** A function $f: X \to Y$ is **continuous** if for every $x \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for every $y \in X$ with $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < \varepsilon$.

**Define:** A function $f: X \to Y$ is **uniformly continuous** if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. for every $x, y \in X$ with $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < \varepsilon$.

**Define:** A function $f: X \to Y$ is **Lipschitz continuous** if $\exists L \geq 0$ s.t. for every $x, y \in X$, $\|f(x) - f(y)\| \leq L \|x - y\|$.

**Remark:** Continuous implies no jumps.
Uniformly continuous implies bounds on the rate of change depend only on the distance b/t points, not the specific pts.
Lipschitz continuous implies the rate of change is strictly bounded everywhere multiplicatively (or at least locally everywhere)

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bounded everywhere multiplicatively (or at least locally everywhere in a neighborhood).

On a compact set (e.g. a closed interval),

Continuously differentiable $\subset$ Lipschitz $\subset$ Uniformly continuous $=$ continuous.

So the thm is applicable generally, but we can think of $f(t, x)$ as having continuous derivatives, because in that special case, all the uniformities we care about are satisfied.

In our proof we assume Lipschitz continuity for simplicity.

Proof. Next time.